

Complex linear spaces and normed linear spaces (Examples)

Example 1: Let N be a real normed linear space and suppose $f(x) = 0$ for all $f \in N^*$. Show that $x = 0$.

Proof: — Suppose $x \neq 0$. Then, there exists $f \in N^*$ such that $f(x) = \|x\| > 0$ which contradicts the hypothesis that $f(x) = 0$ for all $f \in N^*$. Hence we must have $x = 0$.

Example 2: Let M be a closed linear subspace of a normed linear space N and let x_0 be a vector not in M . If d is the distance from x_0 to M , show that there exists a functional $f \in N^*$ such that $f(M) = \{0\}$, $f(x_0) = d$ and $\|f\| = 1$.

Solution: — First we have by definition:

$$d = \inf \{ \|x_0 - x\| : x \in M \}.$$

Since M is closed and $x_0 \notin M$, $d > 0$.

Now consider the subspace $M_0 = \{x + \alpha x_0 : x \in M, \alpha \text{ real}\}$ spanned by M and x_0 . Since $x_0 \notin M$, the representation of each vector y in M_0 in the form $y = x + \alpha x_0$ is unique.

Define the map f_0 on M_0 by

$$f_0(y) = \alpha d$$

where $y = x + \alpha x_0$ and d as in the hypothesis. Because of the uniqueness of y , the mapping f_0 is well defined. It is clear that f_0 is linear on M_0 . Also $f_0(x_0) = f_0(1 \cdot x_0) = 1 \cdot d = d$ and if $m \in M$, then $f_0(m) = f_0(m + 0 \cdot x_0) = 0 \cdot d = 0$ so that $f_0(M) = \{0\}$. We now prove that $\|f_0\| = 1$.

We have $\|f_0\| = \sup \left\{ \frac{|f_0(y)|}{\|y\|} : y \in M_0, y \neq 0 \right\}$.

$$= \sup \left\{ \frac{|f_0(x + \alpha x_0)|}{\|x + \alpha x_0\|} : x \in M, \alpha \in \mathbb{R}, \alpha \neq 0 \text{ or } \alpha = 0 \right\}$$

$$= \sup \left\{ \frac{|\alpha d|}{\|x + \alpha x_0\|} : x \in M, \alpha \in \mathbb{R}, \alpha \neq 0 \right\}.$$

(∵ $f_0(y) = 0$ when $\alpha = 0$)

$$= \sup \left[\frac{d}{\|x_0 + \frac{x}{\alpha}\|} : x \in M, \alpha \in \mathbb{R}, \alpha \neq 0 \right]$$

[∵ $d > 0, |\alpha d| = d|\alpha|$]

$$= d \sup \left\{ \frac{1}{\|x_0 - z\|} : z = -x/\alpha \in M \right\}$$

$$= d \left[\inf \{ \|x_0 - z\| : z \in M \} \right]^{-1} = d \cdot \frac{1}{d} = 1 \text{ by (1)}$$

Thus f_0 is a linear functional on M_0 such that

$f_0(M) = \{0\}$, $f_0(x_0) = d$ and $\|f_0\| = 1$. (2)
 Hence by the Hahn-Banach theorem there exists
 a functional F on the whole space N such that
 $F(x) = f_0(x) \forall x \in M_0$ and $\|F\| = \|f_0\|$.
 It follows from (2) that
 $F(M) = \{0\}$, $F(x_0) = d$ and $\|F\| = 1$.
Proved!

Example-03. Prove that a normed linear space is separable
 if its conjugate (dual) space is separable.

Solution: — Let N be a normed linear space whose
 conjugate space N^* is separable. Consider the set

$$S = \{f : f \in N^*, \|f\| = 1\}.$$

Since every subspace of a metric space is separable,
 S must be separable. Hence by definition of
 separability, S contains countable dense subset, say

$$A = \{f_1, f_2, \dots, f_n, \dots\}.$$

Since each $f_n \in S$, we have $\|f_n\| = 1$ for all n .

Since $\|f_n\| = \sup \{|f_n(x)| : \|x\| = 1\}$, for each n ,
 there must exist some vector x_n with $\|x_n\| = 1$ such that
 $|f_n(x_n)| > \frac{1}{2}$.

(If such x_n did not exist, this would contradict the
 fact that $\|f_n\| = 1$.)

Let M be the closed linear subspace
 in N generated by the sequence $\langle x_n \rangle$. We assert
 that $M \neq N$. Suppose $M = N$ and let $x_0 \in N - M$

i.e. $x_0 \notin M$ by above example, there exists
 an $F \in N^*$ such that

$$\|F\| = 1, F(x_0) \neq 0 \text{ and } F(x) = 0 \text{ if } x \in M$$

Since $\|F\| = 1$, $F \in S$ and since each $x_n \in M$, we have
 $F(x_n) = 0$ ($n = 1, 2, 3, \dots$). Now

$$\begin{aligned}
 \frac{1}{2} < |f_n(x_n)| &= |(f_n(x_n) - F(x_n)) + F(x_n)| \\
 &\leq |f_n(x_n) - F(x_n)| + |F(x_n)| \\
 &= |(f_n - F)(x_n)| \quad (\because F(x_n) = 0) \\
 &\leq \|f_n - F\| \|x_n\| = \|f_n - F\| \quad (\because \|x_n\| = 1)
 \end{aligned}$$

Thus $\|f_n - F\| \geq \frac{1}{2}$ for all n .

Now, since A is dense in S , every point of S is an adherent point of A so that each sphere centred at arbitrary $f \in S$ must contain a point of A . But the open sphere $\{f: \|f - F\| < \frac{1}{2}\}$ centred at $F \in S$ contains no point of A . ~~by~~ We thus arrive at a contradiction and so we must have $M = N$. It then follows that the set of all linear combinations of the x_n 's whose coefficients are rational or if N is complex have rational real and imaginary parts, constitute a countable set everywhere dense in N and consequently N is separable.

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